

# Simultaneous Reconstructions of Absorption Density and Wave Speed with Photoacoustic Measurements <sup>\*</sup>

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## Abstract

In this paper we propose an approach for *simultaneous* identification of the *absorption density* and the *speed of sound* by photoacoustic measurements. Experimentally our approach can be realized with sliced photoacoustic experiments. The mathematical model for such an experiment is developed and exact reconstruction formulas for both parameters are presented.

## 1 Introduction

In this paper we propose an approach for *simultaneous* identification of the *absorption density* and the *speed of sound* by photoacoustic measurements.

In standard photoacoustic experiments the object is *uniformly* illuminated by a short electromagnetic impulse. Recently *sectional* photoacoustic imaging techniques [17, 22, 9, 10] and according reconstructions techniques [4] have been developed. In sectional experiments thin hyperbola shaped like regions of the specimen are imaged (see Figure 1) sequentially. Experimentally, one can perform sectional photoacoustic imaging by focused illumination combined with focusing detectors. The focused illumination is achieved by using cylindrical lenses in front of the object. Moreover, contemporary focusing ultrasonic detectors have a spherical or cylindrical shape, thus the detector surface plays the role of the acoustic lens. The generated ultrasonic wave is refracted by a suitable acoustic lens such that out-of-plane (center of the hyperbola shaped regions) signals are generally weak and can be neglected. Thus essentially only signals emerging from the imaging plane are collected at the detector. This justifies

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that the illumination can be assumed restricted to a single plane (line in 2D). However, such a model requires a low scattering coefficient of the sample (which model organism specimens like the Zebra fish have). Opposed to *sectional 3D*

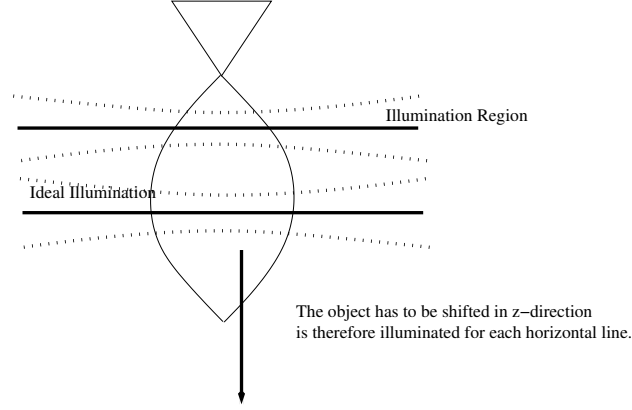


Fig. 1: Conventional sectional photoacoustic imaging.

*photoacoustic imaging*, where stacks of complementary two-dimensional projection images are produced, the proposed approach for simultaneous imaging consists in performing overlapping sliced imaging by rotation and translation of the specimen. This, also generates enough data for reconstructing the two independent parameter functions, speed of sound and absorption density, (see Figure 2). By now the approach presented here is by far from being experi-

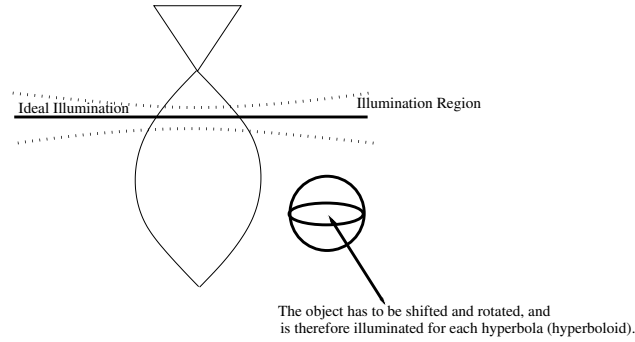


Fig. 2: Sectional imaging in all directions produces enough data to reconstruct both imaging functions.

mentally economical in the sense that a quick dimension analysis shows that we acquire much too many data - in  $\mathbb{R}^3$  we reconstruct two 3-dimensional functions from eventually six dimensional data. It is the goal of this paper, however, not to present the most economical approach, but to show that it is possible to derive exact reconstruction formulas for both imaging parameter functions.

The literature on reconstruction formulas and back-projection algorithms for photoacoustic imaging is vast. Wang et al. developed reconstruction formulas for cylindrical, spherical, and planar measurement geometries in a series of papers [27, 26, 25, 28], and recently many more algorithms based on reconstruction

formulas have been developed (see the survey [15]). As it becomes transparent below we can make use of reconstruction formulas from photoacoustics—essentially we make extensive use of inversion formulas for the spherical mean operator. Exact reconstruction formulas for the wavespeed function in ultrasound reflectivity tomography are based on the *Born approximation* to the wave equation, which is valid for moderately varying speed of sound. Reconstruction formulas for ultrasound reflectivity tomography have already been derived by Norton & Linzer [18, 19] and are also based on inversion formulas for the spherical mean operator. The possibility of exact inversion in both fields supports to derive exact inversion formulas for both parameters.

The proposed method is a hybrid and quantitative imaging methods (see [3, 14, 2]) for some recent surveys. Most closely related to our approach is the work of Stefanov and Uhlmann [24] which presented a photoacoustic experiment for recovering either the sound speed or absorption density. A substitute to our work is also the recovery of the absorption density function for inhomogeneous wave speed [1, 11, 23].

The reconstruction formulas for simultaneous imaging utilize techniques from reflectivity imaging and photoacoustic imaging. In the current state of research we can provide exact reconstruction formulas, but when we try to be economical, i.e., using less slicing experiments, we would require reconstruction formulas, which have not been developed so far. This is highlighted in Section 2. The outline of this paper is as follows: In Section 2 we discuss the mathematical model and discuss the non-economicality of proposed model, which leaves room for further improvement of the results. In Section 3 we transform the model into the Fourier domain, from which in Sections 3.2, 3.1 exact reconstruction formulas are derived in 2D, 3D, respectively. The Appendix provides the exact definitions used in this paper.

## 2 Model

In  $\mathbb{R}^n$ , we consider the following Cauchy problem for the wave equation:

$$\begin{aligned} \frac{1}{c^2} \partial_{tt} \tilde{u} - \Delta \tilde{u} &= 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_{>0}, \\ \tilde{u}(x, 0) &= f(x) \delta_{r,\theta}(x) \quad \text{in } \mathbb{R}^n, \\ \partial_t \tilde{u}(x, 0) &= 0 \quad \text{in } \mathbb{R}^n. \end{aligned}$$

where  $c = c(x)$  denotes the speed of sound and  $\delta_{r,\theta}(x) = \delta(\text{dist}(x, E(r, \theta)))$  where  $E(r, \theta)$  is the  $(n - 1)$ -dimensional hyperplane with distance  $r$  from the origin and orientation  $\theta \in S^{n-1}$ . This is the photoacoustic equation with sliced illumination in the plane  $E(r, \theta)$ . It is the ultimate aim to reconstruct  $c$  and  $f$  from measurements of  $\tilde{u}$  on some hypersurface  $\Gamma$  (see below).

We consider the Born approximation; that is, we expand  $\tilde{u}$  formally with respect to the contrast function  $q := 1/c^2 - 1$  and consider only terms of order at most  $q$ . This leads to the decomposition  $\tilde{u} \approx u + v$  where  $u = u^{r,\theta}$  is the solution of

the wave equation

$$\begin{aligned}\partial_{tt}u - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_{>0}, \\ u(x, 0) &= f(x) \delta_{r,\theta}(x) \quad \text{in } \mathbb{R}^n, \\ \partial_t u(x, 0) &= 0 \quad \text{in } \mathbb{R}^n,\end{aligned}$$

and  $v = v^{r,\theta}$  solves

$$\begin{aligned}\partial_{tt}v - \Delta v &= -q(x) \partial_{tt}u \quad \text{in } \mathbb{R}^n \times \mathbb{R}_{>0}, \\ v(x, 0) &= 0 \quad \text{in } \mathbb{R}^n, \\ \partial_t v(x, 0) &= 0 \quad \text{in } \mathbb{R}^n.\end{aligned}$$

The dependence of  $v$  on  $r, \theta$  is through the source function  $u$ .

It is our goal to reconstruct  $q$  and  $f$  from measurements of

$$m^{r,\theta}(x, t) = u^{r,\theta}(x, t) + v^{r,\theta}(x, t), \quad (x, t) \in \Gamma \times (0, T), \quad (2.1)$$

where  $\Gamma$  is a  $(n-1)$ -dimensional hypersurface in  $\mathbb{R}^n$ . Currently everything is fixed to complete measurements on the *whole* surface  $\Gamma$ .

We make the following assumptions:

1.  $\Gamma = \partial B$  for some open and connected set  $B \subset \mathbb{R}^n$  with smooth boundary. As typically in photoacoustics, we consider  $B$  to be a sphere, a circle, or a halfplane.
2.  $f$  is the sum of some known initial distribution  $f_0$  with compact support and some unknown term  $f_1$  to be determined.
3. The supports of  $f_1$  and  $q$  are both contained in  $\Omega$  for some bounded domain  $\Omega \subset B$ ,
4.  $f_0, f_1$  and  $q$  are smooth and  $f(x) = f_0(x) + f_1(x) \neq 0$  for all  $x \in \overline{B}$ .

## Dimensionality Analysis

In the  $n$ -dimensional setting, we record measurements (cf. (2.1)) for every  $(x, t) \in \Gamma \times (0, T)$  and every  $(r, \theta) \in (0, \infty) \times S^{n-1}$ . That is, the recorded data are  $2n$ -dimensional. The data to be recovered are  $f_1$  and  $q$ , which are two  $n$ -dimensional functions. In general we think that we record too many data for the purpose of reconstructing  $f_1$  and  $q$  – however, since we rely on Radon transforms techniques for exact inversion, we accept this disadvantage for the mathematical studies.

## 3 Fourier Reconstruction Formulas

In this section we derive exact reconstruction formulas for  $f_1$  and  $g$ . First we derive a general inversion formula, which is then evaluated differently in 2D and 3D.

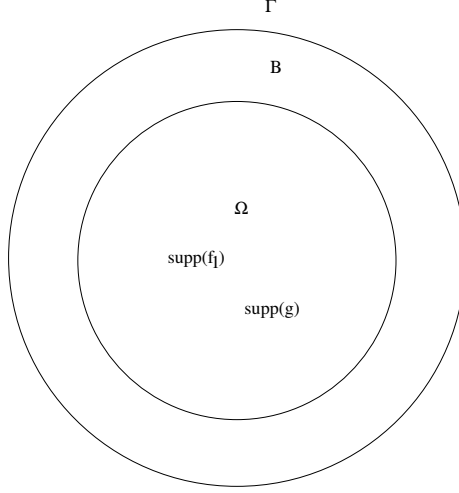


Fig. 3: Schematic representation of the domains and supports.

In the following we omit the superscripts  $(r, \theta)$  for the sake of convenience of notation. Let

$$\hat{u}(x, k) = \mathcal{F}u(x, k) = \int_0^\infty u(x, t) e^{ikt} dt, \quad x \in \mathbb{R}^n, \quad k \in \mathbb{R},$$

be the Fourier transform - note that  $u$  vanishes for  $t < 0$ . We assume that  $u$  is bounded in  $\mathbb{R}^n \times \mathbb{R}_{>0}$  such that  $u(x, \cdot) \in L^2(0, \infty)$  for every  $x \in \mathbb{R}^n$ . Then we note that  $\hat{u}$  has the following properties (by the theorem of Paley-Wiener):

- (a)  $u(x, \cdot)$  has a holomorphic extension into  $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$  for all  $x \in \mathbb{R}^n$ ,
- (b) for every  $x \in \mathbb{R}^n$  it holds that  $\int_{-\infty}^\infty |\hat{u}(x, k_1 + ik_2) - \hat{u}(x, k_1)|^2 dk_1 \rightarrow 0$  as  $k_2$  tends to zero,
- (c)  $\hat{u}(\cdot, k)$  is bounded in  $\mathbb{R}^n$  for all  $k \in \mathbb{C}_+$ .

We have

$$\begin{aligned} \hat{u}(x, k) &= \mathcal{F}u(x, k) = \int_0^\infty u(x, t) e^{ikt} dt \\ &= \frac{1}{ik} \left[ u(x, t) e^{ikt} \Big|_{t=0}^\infty - \int_0^\infty \partial_t u(x, t) e^{ikt} dt \right] \\ &= -\frac{1}{ik} u(x, 0) - \frac{1}{(ik)^2} \left[ \partial_t u(x, t) e^{ikt} \Big|_{t=0}^\infty - \int_0^\infty \partial_{tt} u(x, t) e^{ikt} dt \right] \\ &= -\frac{1}{ik} u(x, 0) - \frac{1}{k^2} \Delta \hat{u}(x, k), \end{aligned}$$

or in other words

$$\Delta \hat{u}(x, k) + k^2 \hat{u}(x, k) = ik u(x, 0). \quad (3.1)$$

Moreover, we have

$$\begin{aligned}
\hat{v}(x, k) &= \mathcal{F}v(x, k) = -\frac{1}{k^2} \int_0^\infty v_{tt}(x, t) e^{ikt} dt \\
&= -\frac{1}{k^2} \int_0^\infty [\Delta v(x, t) - q(x) \partial_{tt} u(x, t)] dt \\
&= -\frac{1}{k^2} \Delta \hat{v}(x, k) + \frac{q(x)}{k^2} \int_0^\infty \partial_{tt} u(x, t) e^{ikt} dt \\
&= -\frac{1}{k^2} \Delta \hat{v}(x, k) + \frac{q(x)}{k^2} \Delta \hat{u}(x, k),
\end{aligned}$$

and thus

$$\begin{aligned}
\Delta \hat{v}(x, k) + k^2 \hat{v}(x, k) &= q(x) \Delta \hat{u}(x, k) \\
&= -k^2 q(x) \hat{u}(x, k) + ik q(x) u(x, 0).
\end{aligned}$$

Let  $\Phi_k$  be the (radiating) fundamental solution of the Helmholtz equation  $\Delta \hat{u} + k^2 \hat{u} = 0$  in  $\mathbb{R}^n$  for  $k \in \mathbb{C}$  with  $\Im z \geq 0$ ; that is, in particular for  $n = 2, 3$ ,

$$\Phi_k(x, y) := \begin{cases} \frac{\exp(ik|x-y|)}{4\pi|x-y|} & \text{for } n = 3, \\ \frac{i}{4} H_0^{(1)}(k|x-y|) & \text{for } n = 2, \end{cases} \quad x \neq y,$$

where  $H_0^{(1)}$  denotes the Hankel function of the first kind and order zero. Then *one particular* solution of (3.1) is given by

$$\begin{aligned}
\hat{u}(x, k) &= -ik \int_{\mathbb{R}^n} u(y, 0) \Phi_k(x, y) dy \\
&= -ik \int_{y \in E(r, \theta)} f(y) \Phi_k(x, y) ds(y).
\end{aligned}$$

We note that this particular solution satisfies the properties (a), (b), (c) from the beginning of this section. Any other solution is of the form  $\hat{u}(\cdot, k) + w(\cdot, k)$  where  $w$  satisfies  $\Delta w + k^2 w = 0$  in  $\mathbb{R}^n$ . The requirement that the solution satisfies the properties (a), (b), (c) from the beginning of this section implies that  $w$  vanishes. Indeed, for  $k \in \mathbb{C}_+$  the function  $w(\cdot, k)$  has to be bounded by property (c), therefore its Fourier transform (in the distributional sense) with respect to  $x \in \mathbb{R}^n$  satisfies

$$\mathcal{F}_x w(y, k) [k^2 - |y|^2] = 0 \text{ for all } y \in \mathbb{R}^n$$

which implies  $\mathcal{F}_x w(\cdot, k) = 0$  because  $k^2 - |y|^2$  does not vanish for  $k \in \mathbb{C}_+$ . Therefore, also  $w(\cdot, k) = 0$  in  $\mathbb{R}^n$  for all  $k \in \mathbb{C}_+$  and, by the continuity property (b), also  $w(\cdot, k) = 0$  in  $\mathbb{R}^n$  for all  $k \in \mathbb{C}$  with  $\Im \geq 0$ .

We also have

$$\begin{aligned}
& \hat{v}(x, k) \\
&= k^2 \int_{\mathbb{R}^n} q(y) \hat{u}(y, t) \Phi_k(x, y) dy - ik \int_{\mathbb{R}^n} q(y) f(y) \delta_{r, \theta}(y) \Phi_k(x, y) dy \\
&= -ik^3 \int_{\mathbb{R}^n} q(y) \int_{z \in E(r, \theta)} f(z) \Phi_k(y, z) ds(z) \Phi_k(x, y) dy \\
&\quad - ik \int_{z \in E(r, \theta)} q(z) f(z) \Phi_k(y, z) ds(z) \\
&= -ik \int_{z \in E(r, \theta)} f(z) \left[ k^2 \int_{\mathbb{R}^n} q(y) \Phi_k(y, z) \Phi_k(x, y) dy + q(z) \Phi_k(x, z) \right] ds(z).
\end{aligned}$$

In summary, we have

$$\begin{aligned}
\hat{m}^{r, \theta}(x, k) &= \hat{u}^{r, \theta}(x, k) + \hat{v}^{r, \theta}(x, k) = \hat{u}(x, k) + \hat{v}(x, k) \\
&= -ik \int_{z \in E(r, \theta)} f(z) \left[ k^2 \int_{\mathbb{R}^n} q(y) \Phi_k(y, z) \Phi_k(x, y) dy \right. \\
&\quad \left. + (q(z) + 1) \Phi_k(x, z) \right] ds(z) \\
&= R[(f(\cdot) L(x, \cdot, k))](r, \theta),
\end{aligned}$$

where  $R[f](r, \theta)$  is the  $(n-1)$ -dimensional Radon transform of  $f$  in direction  $(r, \theta)$  and

$$L(x, z, k) = (-ik)^3 \int_{\mathbb{R}^n} q(y) \Phi_k(y, z) \Phi_k(y, x) dy - ik(q(z) + 1) \Phi_k(x, z) \quad (3.2)$$

for  $x \in \Gamma$ ,  $z \in B$ ,  $k \in \mathbb{R}$ ,  $x \neq z$ .

In other words, we have

$$\boxed{\mathcal{F}^{-1}(R^{-1}(\hat{u} + \hat{v}))(x, z, t) = f(z) \check{L}(x, z, t)}, \quad (3.3)$$

for  $x \in \Gamma$ ,  $z \in B$ ,  $k \in \mathbb{R}$ ,  $x \neq z$ .

Therefore, from the knowledge of  $\hat{m}^{r, \theta}(x, k) = \hat{u}(x, k) + \hat{v}(x, k)$  for all  $x \in \Gamma$ ,  $k \in \mathbb{R}$ ,  $r \geq 0$ , and  $\theta \in S^{n-1}$  we can determine

$$f(z) \check{L}(x, z, t) = (f_0(z) + f_1(z)) \check{L}(x, z, t)$$

for all  $x \in \Gamma$ ,  $z \in B$  with  $x \neq z$ , and  $t \geq 0$ .

### 3.1 3D Domain

For  $n = 3$  we recall that

$$\Phi_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \neq y.$$

Thus from (3.2) it follows that

$$L(x, z, k) = (-ik)^3 \frac{1}{16\pi^2} \int_{\mathbb{R}^3} q(y) \frac{e^{ik(|y-z|+|x-y|)}}{|x-y||y-z|} dy - ik(q(z) + 1) \frac{e^{ik|x-z|}}{4\pi|x-z|}.$$

Taking the inverse Fourier transform with respect to  $k$  gives

$$\begin{aligned}
\check{L}(x, z, t) &= \frac{1}{32\pi^3} \partial_{ttt} \left( \int_{\mathbb{R}^3} \frac{q(y)}{|x-y||y-z|} \int_{-\infty}^{\infty} e^{ik(|y-z|+|x-y|-t)} dk dy \right) \\
&\quad + (q(z) + 1) \frac{1}{8\pi^2|x-z|} \partial_t \left( \int_{-\infty}^{\infty} e^{ik(|x-z|-t)} dk \right) \\
&= \frac{1}{32\pi^3} \partial_{ttt} \left( \int_{\mathbb{R}^3} \delta(|y-z|+|y-x|-t) \frac{q(y)}{|z-y||y-x|} ds(y) \right) \\
&\quad - \frac{q(z) + 1}{8\pi^2|x-z|} \delta'(|x-z|-t) .
\end{aligned}$$

Twice Integration with respect to  $t$  gives

$$\begin{aligned}
&\int_0^{s_2} \int_0^{s_1} \check{L}(x, z, \tau) d\tau ds_1 \\
&= \frac{q(z) + 1}{8\pi^2|x-z|} H(s_2 - |x-z|) \\
&\quad + \partial_{s_2} \left( \underbrace{\int_{\mathbb{R}^3} \delta(|y-z|+|y-x|-s_2) \frac{q(y)}{32\pi^3|z-y||y-x|} ds(y)}_{=: \mathcal{N}(x, z, s_2)} \right) ,
\end{aligned}$$

where we used the Heavyside function  $H(\tau) = 1$  for  $\tau > 0$  and  $H(\tau) = 0$  for  $\tau < 0$ . The initial conditions for the second term vanish since for  $t = 0$  the domain of integration (w.r.t.  $y$ ) is empty.

Integrating once more, we get

$$\begin{aligned}
\Psi(x, z, t) &:= \int_0^t \int_0^{s_2} \int_0^{s_1} \check{L}(x, z, \tau) d\tau ds_1 ds_2 \\
&= \frac{q(z) + 1}{8\pi^2|x-z|} \int_0^t H(s_2 - |x-z|) ds_2 + \mathcal{N}[q](x, z, t) .
\end{aligned} \tag{3.4}$$

Now, we note that

$$\int_0^t H(s_2 - |x-z|) ds_2 = (t - |x-z|)^+ := (t - |x-z|) H(t - |x-z|) ,$$

and thus

$$\Psi(x, z, t) = \frac{q(z) + 1}{8\pi^2|x-z|} (t - |x-z|)^+ + \mathcal{N}[q](x, z, t) . \tag{3.5}$$

Both terms vanish for  $t < |x-z|$ .

We recall that  $M(x, z, t) := f(z)\Psi(x, z, t)$  is known from measurement data for all  $x \in \Gamma$ ,  $z \in B$ , and  $t > 0$ . First we take  $z \in B \setminus \Omega$ . Then  $q(z) = 0$  and  $f(z) = f_0(z)$ , thus  $\Psi(x, z, t) = M(x, z, t)/f_0(z)$  is known and

$$\frac{M(x, z, t)}{f_0(z)} - \frac{(t - |x-z|)^+}{8\pi^2|x-z|} = \mathcal{N}[q](x, z, t) .$$



Again, the left hand side is known. Now we let  $z$  tend to  $x$ . The limit on the right hand side exists, thus also on the left hand side, and therefore for all  $x \in \Gamma$  and  $t > 0$  we have

$$\begin{aligned} \lim_{z \rightarrow x} \left[ \frac{M(x, z, t)}{f_0(z)} - \frac{t}{8\pi^2|x-z|} \right] &= \frac{1}{8\pi^3 t^2} \int_{\mathbb{R}^3} \delta(|y-x| - t/2) q(y) ds(y) \\ &= \frac{1}{8\pi} \mathcal{M}_2[q](x, t/2), \end{aligned} \quad (3.6)$$

where  $\mathcal{M}_2[q]$  is the spherical mean operator (see Appendix).

Thus the reconstruction algorithm is as follows:

1. Calculate the product  $f(z)\check{L}(x, z, t)$  from (3.3) for all  $x \in \Gamma$ ,  $z \in B$  and  $t > 0$ . By integration (see (3.4)), this yields the knowledge of  $M(x, z, t) := f(z)\Psi(x, z, t)$  for all  $x \in \Gamma$ ,  $z \in B$  and  $t > 0$ .
2. Solve (3.6) for  $q$  in  $\Omega$  by inverting the spherical mean operator.
3. Compute  $\Psi(x, z, t)$  for all  $x \in \Gamma$ ,  $z \in \Omega$  and  $t > 0$  from (3.5).
4. Finally, compute  $f$  from  $f(z) = M(x, z, t)/\Psi(x, z, t)$  for  $z \in \Omega$ .

**Remark 3.1** The operator  $\mathcal{N}[q](x, z, t)$  is the rotational ellipsoidal mean operator with focal points  $x$  and  $z$ . Thus the integral equation (3.5); that is,

$$\Psi(x, z, t) = \frac{q(z) + 1}{8\pi^2|x-z|} (t - |x-z|)^+ + \mathcal{N}[q](x, z, t),$$

can, for instance, be considered as a fixed point equation for  $q$  involving the ellipsoidal mean operator and can be solved by the fixed point iteration

$$\Psi(x, z, t) = \frac{q_n(z) + 1}{8\pi^2|x-z|} (t - |x-z|)^+ + \mathcal{N}[q_{n-1}](x, z, t), n = 1, 2, \dots$$

*Ellipsoidal mean operators have been studied in John's book [12].*

### 3.2 2D Domain

In  $\mathbb{R}^2$  we recall that the fundamental solution is given by

$$\Phi_k(x, y) = \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x \neq y.$$

We have to compute the inverse Fourier transform of the product  $\Phi_k(x, y) \Phi_k(z, y)$  (see (3.2)) and use the convolution theorem. First we have that

$$\frac{i}{4} \mathcal{F}^{-1}(H_0^{(1)}(\cdot|x-y|))(t) = \begin{cases} \frac{1}{2\pi\sqrt{t^2 - |x-y|^2}}, & t > |x-y|, \\ 0, & t < |x-y|. \end{cases}$$

The convolution for the inverse Fourier transform is given by the formula  $\mathcal{F}^{-1}(fg) = \check{f} * \check{g}$ ; that is,

$$\mathcal{F}^{-1}(fg)(t) = (\check{f} * \check{g})(t) = \int_{-\infty}^{\infty} \check{f}(\tau) \check{g}(t - \tau) d\tau.$$

Therefore,

$$\begin{aligned} & \left(\frac{i}{4}\right)^2 \mathcal{F}^{-1}(H_0^{(1)}(\cdot|x-y)H_0^{(1)}(\cdot|z-y))(t) \\ &= \begin{cases} \frac{1}{(2\pi)^2} \int_{|x-y|}^{t-|y-z|} \frac{1}{\sqrt{\tau^2-|x-y|^2}} \frac{1}{\sqrt{(t-\tau)^2-|z-y|^2}} d\tau, & t - |y-z| > |x-y|, \\ 0, & t - |y-z| < |x-y|, \end{cases} \end{aligned}$$

and thus from (3.2)

$$\begin{aligned} \check{L}(x, z, t) &= \frac{1}{4\pi^2} \partial_{ttt} \left( \int_{\mathbb{R}^2} q(y) \int_{|x-y|}^{t-|y-z|} \frac{1}{\sqrt{\tau^2-|x-y|^2}} \frac{1}{\sqrt{(t-\tau)^2-|z-y|^2}} d\tau dy \right) \\ &\quad + \frac{1}{2\pi} \partial_t \left( \frac{q(z) + 1}{\sqrt{t^2-|x-z|^2}} H(t-|x-z|) \right), \end{aligned} \tag{3.7}$$

for  $t - |y-z| > |x-y|$  where  $H$  denotes again the Heaviside function. Again, as in 3D, both terms vanish for  $t < |x-z|$ .

Letting  $z$  tend to  $x$  (thus  $t > |x-z|$  for  $z$  sufficiently close to  $x$ ) yields

$$\begin{aligned} \check{L}(x, x, t) &= \frac{1}{4\pi^2} \partial_{ttt} \left( \int_{2|y-x|<t} q(y) \int_{|x-y|}^{t-|y-x|} \frac{1}{\sqrt{\tau^2-|x-y|^2}} \frac{1}{\sqrt{(t-\tau)^2-|x-y|^2}} d\tau dy \right) \\ &\quad + \frac{1}{2\pi} \partial_t \left( \frac{1}{t} \right). \end{aligned}$$

Note that  $q(x)$  vanishes for  $x \in \Gamma$ . Using polar coordinates and denoting

$$k(t, r) := \begin{cases} \int_r^{t-r} \frac{1}{\sqrt{\tau^2-r^2}} \frac{1}{\sqrt{(t-\tau)^2-r^2}} d\tau, & t > 2r > 0, \\ 0, & 0 < t < 2r, \end{cases}$$

we therefore get

$$\int_0^{t/2} k(t, r) r \int_0^{2\pi} q\left(x + r \begin{pmatrix} \cos \rho \\ \sin \rho \end{pmatrix}\right) d\rho dr = 2\pi \int_0^{t/2} \mathcal{M}_1[q](x, r) k(t, r) r dr$$

with the one dimensional circular mean operator  $\mathcal{M}_1$ . Using the notations

$$\begin{aligned} \tilde{k}(t, r) &:= r k(2t, r) \quad \text{and} \\ \Phi(x, t) &:= 4\pi^2 \int_0^t \int_0^{s_2} \int_0^{s_1} \check{L}(x, x, \tau) d\tau ds_1 ds_2 - 2\pi(t \log(t) - t), \end{aligned} \tag{3.8}$$

we get, for fixed  $x$ , the Volterra integral equation for  $\mathcal{M}_1[q](x, \cdot)$ :

$$\Phi(x, t) = \int_0^t \tilde{k}(t, r) \mathcal{M}_1[q](x, r) dr, \quad t \in (0, T]. \quad (3.9)$$

We express  $k$  as a complete elliptic integral: We introduce the variable  $\phi \in (-\pi/2, \pi/2]$  by  $\tau = \frac{t}{2} + (\frac{t}{2} - r) \sin \phi$ . Then  $d\tau = (\frac{t}{2} - r) \cos \phi d\phi$  and, with  $t' = t/2$ ,

$$\begin{aligned} \tau^2 - r^2 &= (\tau - r)(\tau + r) \\ &= [(t' - r) + (t' - r) \sin \phi] [(t' + r) + (t' - r) \sin \phi], \\ (t - \tau)^2 - r^2 &= (t - \tau - r)(t - \tau + r) \\ &= [(t' - r) - (t' - r) \sin \phi] [(t' + r) - (t' - r) \sin \phi]. \end{aligned}$$

For the product we conclude that

$$\begin{aligned} &[\tau^2 - r^2][(t - \tau)^2 - r^2] \\ &= [(t' - r)^2 - (t' - r)^2 \sin^2 \phi] [(t' + r)^2 - (t' - r)^2 \sin^2 \phi] \\ &= (t' - r)^2 \cos^2 \phi [(t' + r)^2 - (t' - r)^2 \sin^2 \phi]. \end{aligned}$$

Therefore,

$$\begin{aligned} k(t, r) &= \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\sqrt{(t' + r)^2 - (t' - r)^2 \sin^2 \phi}} \\ &= \frac{2}{t' + r} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \left(\frac{t' - r}{t' + r}\right)^2 \sin^2 \phi}}, \end{aligned}$$

and thus

$$\tilde{k}(t, r) = \rho K(1 - \rho) \quad \text{with} \quad \rho = \frac{2r}{t + r},$$

and where

$$K(\alpha) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}}, \quad |\alpha| < 1,$$

denotes the complete elliptic integral.

Let  $\delta > 0$  such that  $\delta < \text{dist}(\Gamma, \Omega)$ . Since  $\mathcal{M}_1[q](x, \tau) = 0$  for  $\tau \leq \delta$  we can consider the integral equation (3.9) on  $[\delta, T]$ ; that is,

$$\boxed{\Phi(x, t) = \int_{\delta}^t \tilde{k}(t, r) \mathcal{M}_1[q](x, r) dr, \quad t \in [\delta, T].} \quad (3.10)$$

Now we note that  $\tilde{k}(t, t) = K(0) = \pi/2$  and  $\tilde{k} \in C^1(\Delta)$  where  $\Delta = \{(t, r) \in [\delta, T]^2 : r \leq t\}$ . Therefore, we can transform the equation of the first kind into one of the second kind by differentiating (3.10). This yields

$$\boxed{\Phi(x, t) = \frac{\pi}{2} \mathcal{M}_1[q](x, t) + \int_{\delta}^t \frac{\partial}{\partial t} \tilde{k}(t, r) \mathcal{M}_1[q](x, r) dr, \quad t \in [\delta, T].} \quad (3.11)$$

Volterra integral equations of the second kind with smooth kernel and non-vanishing diagonal have a unique solution, see, e.g., [13, Section 3.3].

After calculating the spherical mean operator  $\mathcal{M}_1[q]$ , it can be inverted by using standard inversion formulas, which exist for various geometries: For  $B = B_R^2(0) \subset \mathbb{R}^2$ , analytical reconstruction formulas have been derived by Finch, Haltmeier, Rakesh [6]. For a general domain  $\Omega$ , Kunyansky reduced in [16] the reconstruction problem to the determination of the eigenvalues  $\lambda_k$  and normalized eigenfunctions  $u_k$ ,  $\|u_k\|_2 = 1$ , of the Dirichlet Laplacian  $-\Delta$  on  $\Omega$  with zero boundary conditions. Recently Palamadov [21] presented a general approach leading to reconstruction algorithms for various geometries. For more details see also the Appendix.

Thus the reconstruction algorithm is as follows:

1. Calculate the product  $M(x, z, t) := f(z)\check{L}(x, z, t)$  from (3.3) for all  $x \in \Gamma$ ,  $z \in B$ ,  $t > 0$ . Compute  $\Phi(x, t)$  from (3.8) for all  $x \in \Gamma$  and  $t > 0$ .
2. Solve (3.11) for calculating  $\mathcal{M}_1[q]$ .
3. Invert the spherical mean operator.
4. From (3.7) determine  $\check{L}$  by differentiation.
5. Finally use  $f(z) = M(x, z, t)/\check{L}(x, z, t)$  to reconstruct  $f(z)$  for  $z \in \Omega$ .

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## 4 Appendix

### Notation

$B_R^n(x)$  denotes the ball of radius  $R$  and center  $x$  in  $\mathbb{R}^n$ .

### Bessel Functions

The Bessel functions, Neumann functions and Hankel functions of order zero are defined as

$$\begin{aligned} J_0(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz \sin(\tau)} d\tau \quad \text{for } z \in \mathbb{C}, \quad \text{and} \\ Y_0(z) &= \frac{2}{\pi} [\gamma + \ln(z/2)] J_0(z) - \frac{2}{\pi} \sum_{\ell=0}^{\infty} \frac{a_{\ell+1}}{(\ell!)^2} \left(\frac{z}{2}\right)^{2\ell} \quad \text{for } z \in \mathbb{C} \text{ with } \Im \geq 0, \\ H_0^{(1)} &= J_0 + iY_0, \end{aligned}$$

where  $\gamma$  denotes Euler's constant,  $a_1 = -\gamma$ , and  $a_n = -\gamma + \sum_{\ell=1}^{n-1} \ell^{-1}$  for  $n \geq 2$ . From this we conclude that  $H_0^{(1)}(-x) = \overline{H_0^{(1)}(x)}$  for  $x \in \mathbb{R}$ .

## The Fourier Transform

In this paper we use the following notation:

$$\hat{f}(k) := \mathcal{F}f(k) := \int_{-\infty}^{\infty} f(t) e^{ikt} dt,$$

denotes the Fourier transform of  $f$  and

$$\check{f}(t) := \mathcal{F}^{-1}f(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k) e^{-ikt} dk,$$

the inverse Fourier transform. For  $f \in L^1(\mathbb{R})$  or  $f$  from the Schwarz space  $\mathcal{S}$  they are defined in the classical sense, for  $f \in L^2(\mathbb{R})$  they are defined by extension using the Parseval formula, and for tempered distributions they are defined by duality.

### Examples:

$$\begin{aligned} \hat{\delta}(k) &= \mathcal{F}\delta(k) = \int_{-\infty}^{\infty} e^{ikt} \delta(t) dt \equiv e^{ik0} = 1, \\ \check{\delta}(t) &= \mathcal{F}^{-1}\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikt} \delta(k) dk \equiv \frac{1}{2\pi}. \end{aligned}$$

This shows that

$$\hat{1}(k) = 2\pi\delta(k) \quad \text{and} \quad \check{1}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikt} dk = \frac{1}{2\pi} \delta(t).$$

Furthermore, we have (see [20])

$$\frac{i}{4} H_0^{(1)}(ka) = \int_a^{\infty} \frac{\exp(ikt)}{2\pi\sqrt{t^2 - a^2}} dt \quad \text{for } k > 0.$$

From  $H_0^{(1)}(-x) = \overline{H_0^{(1)}(x)}$  for all  $x \in \mathbb{R}$  we conclude that this formula holds for all  $k \in \mathbb{R}$  with  $k \neq 0$ . The right hand side is a Fourier transform. Therefore,

$$\mathcal{F}^{-1} \left( \frac{i}{4} H_0^{(1)}(a \cdot) \right) = \begin{cases} \frac{1}{2\pi\sqrt{t^2 - a^2}}, & t > a, \\ 0, & t < a. \end{cases}$$

## Radon Transform

In  $\mathbb{R}^n$ ,  $E(r, \theta)$  denotes the  $(n-1)$ -dimensional hyperplane with orientation  $\theta \in S^{n-1}$  and distance  $r$  from origin.

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with support in  $\Omega$ , then the  $(n-1)$ -dimensional Radon transform of  $f$  is defined by

$$R_{r,\theta}(f) = R[f](r, \theta) = \int_{E(r,\theta)} f(x) ds(x).$$

Thus  $R[f]$  is a function from  $\mathbb{R} \times S^{n-1}$  into  $\mathbb{R}$ .

## Spherical Mean Operator

In  $\mathbb{R}^n$  the *spherical mean operator* is defined as follows (see e.g. [5]):

$$\mathcal{M}_{n-1}[u](x, r) := \frac{1}{|S^{n-1}|} \int_{S^{n-1}} u(x + ry) ds(y) \text{ for } x \in \mathbb{R}^n, r > 0.$$

The spherical mean operator can be written as

$$\mathcal{M}_{n-1}[u](x, r) = \frac{1}{r^{n-1}|S^{n-1}|} \int_{\mathbb{R}^n} \delta(|x - z| - r) u(z) dz \text{ for } x \in \mathbb{R}^n, r > 0.$$

In particular

$$\begin{aligned} \mathcal{M}_2[u](x, r) &= \frac{1}{4\pi^2} \int_{S^2} u(x + ry) ds(y) = \frac{1}{4\pi^2 r^2} \int_{\mathbb{R}^3} \delta(|x - z| - r) u(z) dz, \\ \mathcal{M}_1[u](x, r) &= \frac{1}{2\pi} \int_{S^1} u(x + ry) ds(y) = \frac{1}{2\pi r} \int_{\mathbb{R}^2} \delta(|x - z| - r) u(z) dz. \end{aligned}$$

There exist a variety of reconstruction formulas for averages over circular and spherical means:

### in 2D

- For averaged data on the sphere of Radius  $R$ , that is on  $\partial\Omega = \partial B_R^2(0) \subset \mathbb{R}^2$ , analytical reconstruction formulas have been derived by Finch, Haltmeier, Rakesh [6] and read as follows

$$f(\xi) = \frac{1}{2\pi} \Delta_\xi \left( \int_{S^1} \int_0^{2R} r (\mathcal{M}_1[f])(\theta, r) \log|r^2 - |\xi - \theta|^2| dr ds(\theta) \right)$$

and

$$f(\xi) = \frac{1}{2\pi} \int_{S^1} \int_0^2 (\partial_r r \partial_r \mathcal{M}_1[f])(\theta, r) \log|r^2 - |\xi - \theta|^2| dr ds(\theta).$$

- For a general domain  $\Omega$ , Kunyansky reduced in [16] the reconstruction problem to the determination of the eigenvalues  $\lambda_k$  and normalized eigenfunctions  $u_k$ ,  $\|u_k\|_2 = 1$ , of the Dirichlet Laplacian  $-\Delta$  on  $\Omega$  with zero boundary conditions:

$$\begin{aligned} \Delta u_k(\xi) + \lambda_k u_k(\xi) &= 0, & \xi \in \Omega, \\ u_k(\xi) &= 0, & \xi \in \partial\Omega. \end{aligned}$$

Indeed, if  $(\xi, \eta) \mapsto G_{\lambda_k}(|\xi - \eta|)$  is a free-space rotationally invariant Green's function of the Helmholtz equation and  $n(\xi)$  denotes the outer unit normal vector of  $\partial\Omega$  at  $\xi \in \partial\Omega$ , then

$$f(\xi) = 2\pi \sum_{k=0}^{\infty} \tilde{M}_k u_k(\xi),$$

where

$$\tilde{M}_k = \int_{\partial\Omega} \int_0^\infty r \mathcal{M}_1[f](\eta, r) G_{\lambda_k}(r) \langle \nabla u_k(\eta), n(\eta) \rangle dr ds(\eta).$$

### in 3D

We refer to the survey of Finch and Rakesh [8] (see also [7]). In this article three reconstruction formulas are documented, which are

$$\begin{aligned} f(x) &= -\frac{1}{2\pi R_0} \int_{\partial B_{R_0}^3} \frac{\partial_{tt} (t^2(\mathcal{M}_2[f])(x_0, t))|_{t=|x-x_0|}}{|x-x_0|} ds(x_0), \\ f(x) &= -\frac{1}{2\pi R_0} \int_{\partial B_{R_0}^3} \frac{(\partial_t t \partial_t t(\mathcal{M}_2[f]))(x_0, t)|_{t=|x-x_0|}}{|x-x_0|} ds(x_0), \\ f(x) &= -\frac{1}{2\pi R_0} \Delta \left( \int_{\partial B_{R_0}^3} |x-x_0| \mathcal{M}_2[f](x_0, |x-x_0|) ds(x_0) \right). \end{aligned}$$

Here  $x \in B_{R_0}^3$  and  $f$  is compactly supported in this set.

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